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Subharmonic transitions in a bistable oscillator with bimodal periodic excitation

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We analyze the phenomenon of low-frequency signal enhancement in a bistable system excited by a sum of low-frequency and high-frequency harmonic signals. A mechanism alternate to chaotic resonance is discussed. It is shown that a high-frequency signal may generate interwell transitions of subharmonic frequency. If the frequency of the slow signal is equal or close to a subharmonic frequency of the fast signal, then the improvement of the low-frequency constituent in the output spectrum is due to sustained subharmonic resonance.

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I. INTRODUCTION

In this Rapid Communication we discuss a mechanism of low-frequency signal enhancement different from so-called noise-free stochastic resonance. Stochastic resonance is a phenomenon occurring in a system driven by a combination of periodic signals and random noise, in which the periodic component of the output signal becomes enhanced for an optimal nonzero input noise intensity [1]. Noise-free stochastic resonance, also termed chaotic resonance, is produced via extrinsic [2] or intrinsic [3] chaos without random forcing [2–4]. The chaotic dynamics can be tuned, by varying the control parameter, to achieve the maximum of the periodic constituent in the output spectrum. In both cases, the periodic signal alone is assumed weak enough to escape from the domain of attraction of the stable point or orbit. An additional random or chaotic excitation helps in surpassing the threshold and induces hopping between the different stable states. A relative signal enhancement becomes visible in the hopping dynamics.

The passages between the stable states in a planar bistable system are associated with the successive exits and entries through the unperturbed separatrix [Fig. 1(a)]. The periodically driven bistable system demonstrates chaotic and nonchaotic transportation through the separatrix [3]. Chaotic transportation occurs if the system approaches the separatrix with insufficient energy and the Melnikov condition [5] holds. If the trajectory "runs" through the separatrix with a relatively high velocity, escape through the separatrix is nonchaotic.

We consider the equation of motion of the form

$$\ddot{x} + \varepsilon\beta\dot{x} + U'(x) = \varepsilon\sigma \sin \Omega t + \varepsilon\gamma \sin \omega t, \qquad (1)$$

where the small parameter $0 < \varepsilon \ll 1$ characterizes weak nonconservative terms. The potential $U(x)$ is a continuous and twice continuously differentiable even function having maximum $U(0)=0$ at $x=0$ and two symmetric minima $U(c)=U(-c)<0$ at $x=\pm c$ [Fig. 1(b)].

We take

$$U'(x) = -k^2 x + f(x),$$

where $f(0)=0$, $f(-x)=-f(x)$, $|f(x)|\to\infty$ as $|x|\to\infty$, and $|f(x)|/|x|\to 0$ as $|x|\to 0$. Then, we assume $\Omega/\omega \gg 1$. Thus the first and second terms on the right-hand side of Eq. (1) can be interpreted as the fast and slow signals, respectively.

Chaotic resonance generated by harmonic excitation has been studied earlier [4]. This Rapid Communication discusses a mechanism of signal enhancement in a bistable system, alternate to chaotic resonance. We show that a fast signal of frequency Ω may generate periodic interwell transitions of subharmonic frequency Ω/s with a large magnitude, which is close to the distance between the stable states. This process is interpreted as subharmonic resonance. If the frequency of the slow signal $\omega \approx \Omega/s$, then the enhancement of the low-frequency signal can be considered as sustained subharmonic resonance.

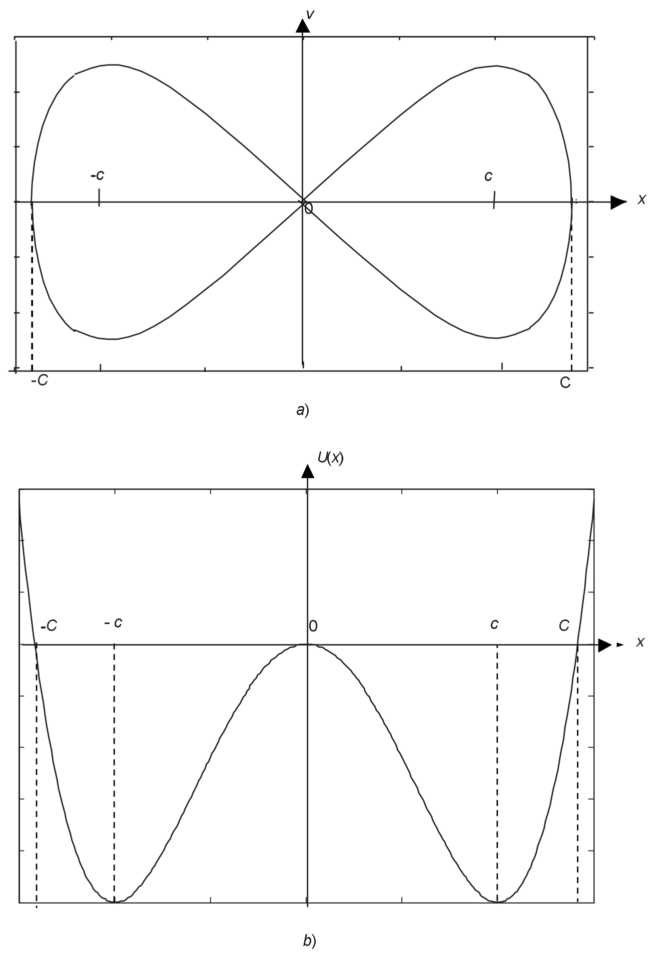

FIG. 1. The unperturbed separatrix (a) and potential (b) of the unperturbed system.

The system with the low-frequency input signal

$$\ddot{x} + \varepsilon\beta\dot{x} + U'(x) = \varepsilon\gamma \sin \omega t \tag{2}$$

exhibits small quasilinear oscillations near the stable point with the amplitude

$$A_\gamma = \frac{\varepsilon\gamma}{[(\lambda^2 - \omega^2)^2 + (\varepsilon\beta\omega)^2]^{1/2}}, \tag{3}$$

where $\lambda=[U''(c)]^{1/2}$ is the frequency of free oscillations in the conservative subsystem ($\varepsilon=0$). If $\omega \ll \lambda$, then $A_\gamma \approx \varepsilon\gamma/\lambda^2$. An additional input signal of frequency $\Omega=\omega s$ may generate subharmonic oscillations of frequency ω with a large magnitude of oscillations [6,7].

Different approximations of the subharmonic solution lead to similar results [6,7]. In this Rapid Communication we use the simple techniques of [7] based on a combination of the Galerkin [8] and harmonic balance methods. In Sec. II, we recall the basic equations. Considering the Duffing system as a typical bistable system, we calculate the amplitude of subharmonic oscillations and prove that subharmonic resonance is associated with transitions between the stable states. The results of the numerical simulation (Sec. III) demonstrate the improvement of the low-frequency signal in the presence of the fast signal.

II. SUBHARMONIC OSCILLATIONS IN SYSTEM (1)

A. Basic equations

Let $T_s=2\pi s/\Omega$ be the period of subharmonic oscillations. Since the nonconservative forces (dissipation and external forcing) in system (1) are small, the subharmonic solution is assumed close to the T_s-periodic oscillations of the conservative subsystem

$$\ddot{x} - k^2 x + f(x) = 0. \tag{4}$$

This assumption implicitly presupposes the resonance mode of subharmonic oscillations in system (1). Let $x_0(t)$ be a known T_s-periodic solution of Eq. (4). Then the approximate T_s-periodic solution of Eq. (1) is sought as [7]

$$x(t,\varphi) = x_0(t+\varphi) + \varepsilon + \cdots , \tag{5}$$

where the parameter φ determines the phase shift between the free and forced oscillations and small correction terms of order ε are omitted. Omitting the detailed derivation (see, e.g., [7]), we note that the equation for the phase φ is reduced to the equation of the balance of work: the work done by the nonconservative forces along the locked periodic orbit is equal to zero. This equality can be written as

$$P(\varphi) = \int_0^{T_s} [\sigma \sin \Omega t + \gamma \sin(\Omega t/s) - \beta\dot{x}_0(t+\varphi)]\dot{x}_0(t+\varphi)dt = 0. \tag{6}$$

The function $P(\varphi)$ expresses the work performed by the nonconservative forces over the period T_s; if $P(\varphi)<0$, the loss of energy due to dissipation exceeds the energy acquired due to external forcing over the period T_s and $x(T_s,\varphi)<x(0,\varphi)$; on the contrary, if $P(\varphi)>0$, then $x(T_s,\varphi)>x(0,\varphi)$; finally, $P(\varphi)=0$ on the periodic solution.

We now find the generating solution $x_0(t)$. Formally, a T_s-periodic solution may be presented as an infinite Fourier series. The approximate subharmonic solution is sought as the sum of the principal harmonic of frequency Ω and the substantial subharmonic of frequency $\omega=\Omega/s$ [6,7]:

$$x_0(t) \approx A \sin(\Omega t/s) + B \sin \Omega t, B \ll A. \tag{7}$$

As the nonlinear term in Eq. (1) is odd, the principal harmonic produces only odd subharmonics—that is, $s=2k+1, k\geqslant 1, s\geqslant 3$, where k and s are integers [6]. The amplitudes A and B are found approximately by the harmonic balance method [7]. To this aim, we substitute expression (6) into the function $f(x)$ and construct the Fourier series of the function $f(A \sin(\Omega t/s)+B \sin \Omega t)$. Considering only the substantial terms, we obtain a truncated Fourier series in the form

$$f(A \sin(\Omega t/s) + B \sin \Omega t) = q_1(A,B)\sin \Omega t + q_s(A,B)\sin(\Omega t/s) + (\text{insubstantial harmonics}), \tag{8}$$

where the Fourier coefficients q_r, $r=1$, s, are calculated as

$$q_r(A,B) = \frac{1}{\pi}\int_0^{2\pi} f(A \sin \phi + B \sin(s\phi))\sin(r\phi)d\phi. \tag{9}$$

Substituting expansions (7) and (8) into Eq. (4), we obtain the equation

$$[q_1(A,B) - (k^2+\Omega^2)B]\sin \Omega t + \{q_s(A,B) - [k^2 - (\Omega/s)^2]A\}\sin(\Omega t/s) = 0, \tag{10}$$

which must be valid at each moment t. This implies that the coefficients before $\sin \Omega t$ and $\sin(\Omega t/s)$ are equal to zero—that is,

$$q_1(A,B) - (k^2+\Omega^2)B = 0,$$
$$q_s(A,B) - [k^2 + (\Omega/s)^2]A = 0. \tag{11}$$

Assuming $\omega=\Omega/s \ll k$ and considering the condition $B\ll A$, we obtain from Eqs. (11)

$$q(A) - k^2 A = 0,$$
$$q_1(A,B) - (k^2+\Omega^2)B = 0. \tag{12}$$

This implies that the leading-order approximation of the subharmonic amplitude A is independent of the amplitude B and the excitation frequency Ω.

Finally, we obtain the equation for the phase φ. Inserting solution (7) into integrand (6) and considering $\omega=\Omega/s$, we find

$$P(\varphi) = -\beta\Omega[A^2 + (sB)^2]/s + s\sigma B \sin(\Omega\varphi) + \gamma A \sin(\omega\varphi) = 0. \tag{13}$$

Despite its simplicity, the harmonic balance approximation of the precise periodic solution is highly accurate provided

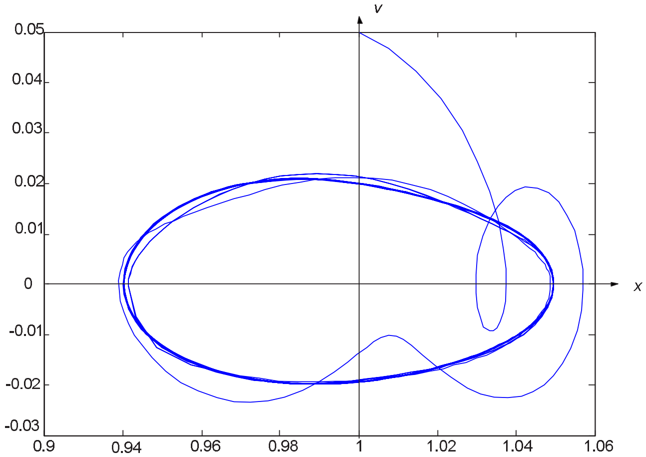

FIG. 2. (Color online) Small oscillations in system (18).

the function $f(x)$ is smooth enough and the nonconservative forces parametrized by the parameter ε are small. Rigorous estimates can be found in [9].

B. Subharmonic transitions in the Duffing equation

The harmonic balance coefficients are integral characteristics slightly depending on the shape of the function f. This allows investigation of subharmonic oscillations in the Duffing system as a typical phenomenon. Using a proper scaling, we obtain the Duffing potential in the form

$$U(x) = -x^2/2 + x^4/4. \tag{14}$$

The equations of motion are written as

$$\ddot{x} + \varepsilon\beta\dot{x} - x + x^3 = \varepsilon\sigma \sin \Omega t + \varepsilon\gamma \sin \omega t, \quad \omega = \Omega/s. \tag{15}$$

From Eqs. (9) and (12) we find

$$q(A) = 3A^3/4, \quad -k^2 + 3A^2/4 = 0, \quad A = 2/\sqrt{3}. \tag{16}$$

Obviously $A \gg A_\gamma$. We now show that subharmonic oscillations correspond to periodic transitions between the stable states. The stable states $\pm c$ of the unperturbed system with potential (14) are found from the equation $U'(x)=0$, which gives $c=1$. The points $\pm C$ of the potential function intersection with the x axis (Fig. 1) are defined by the condition $U(x)=0$—that is, $C=\sqrt{2}$. It follows from Eqs. (16) that $A=2/\sqrt{3}$ and $c<A<C$. This implies that the magnitude $2A$ of subharmonic oscillations is more than the distance between two stable points $\pm c$ but less than the distance between two external points $\pm C$. Thus the trajectory of subharmonic oscillations corresponds to the transitions between two stable positions but it does not escape to the domain of large oscillations outside the separatrix.

Since weak periodic forcing generates subharmonic oscillations of large magnitude, subharmonic transitions can be associated with subharmonic resonance. It follows from Eq. (13) that subharmonic resonance may exist even in the absence of a slow signal ($\gamma=0$). A simple example demonstrates that the slow signal sustains subharmonic transitions.

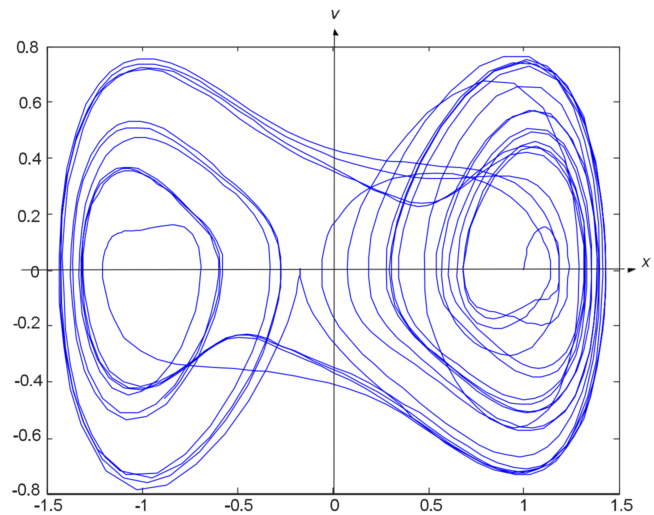

FIG. 3. (Color online) Interwell transitions in system (18).

Let $\sin(\omega\varphi)>0$, $0<\omega\varphi<\pi$. It is easy to see that the additional positive term $\gamma A \sin(\omega\varphi)$ in Eq. (13) increases an admissible level of dissipation, allowing subharmonic transitions. Subharmonic oscillations in the presence of a slow signal can thus be interpreted as sustained subharmonic resonance.

Subharmonic resonance is visible if the period of subharmonic oscillations is comparable with typical time constants of the system. The higher subharmonic resonances exist and may be stable but the domains of stability are negligibly small [6].

III. NUMERICAL SIMULATION

The simulation results of [4] confirmed the presence of chaotic transportation and chaotic resonance in system (15) with bimodal excitation with frequencies $\Omega=1.1$, $\omega=0.0632$, and $s=\Omega/\omega\approx 17$. The frequency $\Omega=1.1$ was chosen corresponding to the maximum of the Melnikov shape factor [4] and thus yielding the most intensive chaotic transportation. The task is to demonstrate the existence of nonchaotic subharmonic transitions in system (15).

The pronounced subharmonic resonance in a system with cubic nonlinearity corresponds to $s=3$; the domains of existence and stability of higher subharmonic resonances are negligibly small and embedded in the "chaotic sea" [5,6]. This makes investigation of the subharmonic resonance of frequency $\Omega/17$ meaningless. We simulate the dynamics of system (15) with the parameters of the same range as in [4]—namely, $\varepsilon\beta=0.316$, $\Omega=1.1$, $\varepsilon\sigma=0.3$, and $\varepsilon\gamma=0.1$—but we take a slow signal of frequency $\omega=\Omega/3$. Computation has been performed with MATLAB software using 32 significant digits.

The motion of the system,

$$\ddot{x} + \varepsilon\beta\dot{x} - x + x^3 = 0.1 \sin\left(\frac{1.1}{3}t\right), \tag{17}$$

is close to small linear oscillations of frequency $\omega=1.1/3\approx 0.367$ near the stable point $x=1, v=0$ (Fig. 2).

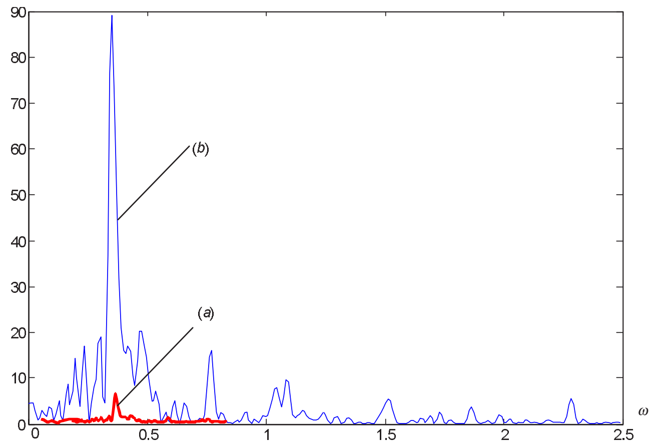

FIG. 4. (Color online) The output spectra of systems (17) (a) and (18) (b) with peaks of $\omega=0.367$.

Adding an additional fast signal of frequency $\Omega=1.1$, we obtain the system with bimodal excitation:

$$\ddot{x} + \varepsilon\beta\dot{x} - x + x^3 = 0.3\sin(1.1t) + 0.1\sin\left(\frac{1.1}{3}t\right). \quad (18)$$

Figure 3 demonstrates the appearance of the interwell transitions in system (18). The averaged magnitude of oscillations slightly exceeds the theoretical value $A=2/\sqrt{3}\approx 1.155$, but in a qualitative sense, the transitions dynamics is consistent with the theoretical description. The trajectory of subharmonic oscillations overlaps the stable positions but it does not escape to the outer domain. The enhancement of the low-frequency component in the output spectrum is shown in Fig. 4.

IV. CONCLUSION

We have demonstrated that the phenomenon of low-frequency signal enhancement in a bistable oscillator with bimodal periodic excitation is not directly associated with chaotic resonance. The high-frequency component of the periodic excitation may generate periodic interwell transitions of subharmonic frequency with the magnitude exceeding the distance between the stable states. This process is interpreted as subharmonic resonance. Assuming the frequency of a slow signal equal or close to the transitions frequency, we have treated the slow signal enhancement as a demonstration of sustained subharmonic resonance.

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